

PERFECT SET THEOREMS FOR EQUIVALENCE RELATIONS WITH I -SMALL CLASSES

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ABSTRACT. A classical theorem due to Mycielski states that an equivalence relation E having the Baire property and meager equivalence classes must have a perfect set of pairwise inequivalent elements. We consider equivalence relations with I -small equivalence classes, where I is a proper σ -ideal, and ask whether they have a perfect set of pairwise inequivalent elements. We give a positive answer for E universally Baire. We show that the answer for $E \Delta_2^1$ is independent of ZFC , and find set theoretic assumptions equivalent to it when I is the countable ideal.

For equivalence relations which are Σ_2^1 and with meager classes, we show that a perfect set of pairwise inequivalent elements exists whenever a Cohen real over $L[z]$ exists for any real z – which strengthens Mycielski’s theorem.

A few comments are made about σ -ideals generated by Π_1^1 and orbit equivalence relations.

1. INTRODUCTION

We say that an equivalence relation E on a Polish space X *has perfectly many classes* if there is a perfect set $P \subseteq X$ such that all elements of P are pairwise inequivalent.

Two classical theorems due to Mycielski claim:

Theorem 1.1. *If E is an equivalence relation that has the Baire property, and all E -classes are meager, then E has perfectly many classes.*

Theorem 1.2. *If E is an equivalence relation that is Lebesgue measurable, and all E -classes are null, then E has perfectly many classes.*

This paper is about equivalence relations with small classes, and investigate the cases in which such equivalence relations must have many classes, namely, perfectly many classes. We will restrict our discussion to equivalence relations which are not more complicated than Σ_2^1 or Π_2^1 . However, we would like to consider a much wider class of notions of “small” sets:

Definition 1.3. Given a σ -ideal I on a Polish space X , we say that $A \subseteq X$ is I -positive if $A \notin I$, and an I -small set if $A \in I$. We denote by \mathbb{P}_I the partial order of Borel I -positive sets ordered by inclusion. We say that I is *proper* if \mathbb{P}_I is a proper forcing notion.

We can now state the main problem discussed in this paper:

Problem 1.4. Let I be a proper σ -ideal and E a Σ_α^1 , Π_α^1 or Δ_α^1 equivalence relation with I -small classes. Does E have perfectly many classes?

1.1. The results of this paper. To make statements easier, we fix the following notation:

Definition 1.5. For I a σ -ideal, $PSP_I(\Sigma_n^1)$ (for “Perfect Set Property”) is the following statement:

“If E is a Σ_n^1 equivalence relation with I -small classes then E has perfectly many classes”.

In section 2 we prove the following:

Theorem 1.6. *Let E be a universally Baire equivalence relation, and I a proper σ -ideal. If all E -classes are I -small, then E has perfectly many classes.*

Corollary 1.7. *Let E be an analytic equivalence relation, and I a proper σ -ideal. If all E -classes are I -small, then E has perfectly many classes. In other words, for any proper σ -ideal I , $PSP_I(\Sigma_1^1)$ is true.*

We could have stated the same for E coanalytic, but that will follow immediately of Silver's theorem on coanalytic equivalence relations. Note that some assumption on I has to be made: given E analytic with uncountably many Borel classes but not perfectly many classes, let I_E be the σ -ideal generated by the equivalence classes. Then all E classes are I_E -small, but E does not have perfectly many classes. Indeed, such I is never proper.

In section 3 we expand our discussion to the class of Δ_2^1 equivalence relations. The case of provably Δ_2^1 equivalence relations is no different then the analytic case, since those are universally Baire. But in the case of a general Δ_2^1 equivalence relation, problem 1.4 is independent of ZFC :

Theorem 1.8. *Let I be a proper σ -ideal, and assume Π_3^1 - \mathbb{P}_I -generic absoluteness. Then $PSP_I(\Delta_2^1)$.*

Theorem 1.9. [4] *If $\mathbb{R} = \mathbb{R}^{L[z]}$ for some $z \in \mathbb{R}$, then for any σ -ideal I , $\neg PSP_I(\Delta_2^1)$.*

We use the above to completely solve the problem for the countable ideal and Δ_2^1 equivalence relations:

Theorem 1.10. (countable ideal) *The following are equivalent:*

- (1) $PSP_{countable}(\Delta_2^1)$.
- (2) For z real, $\mathbb{R}^{L[z]} \neq \mathbb{R}$.

In section 4 we consider Σ_2^1 and Π_2^1 equivalence relations for the case of the meager ideal:

Theorem 1.11. *If for any real z there is a Cohen real over $L[z]$ then*

$$PSP_{meager}(\Sigma_2^1)$$

and

$$PSP_{meager}(\Pi_2^1 \text{ with Borel classes}).$$

That strengthens Mycielski's theorem 1.1: if there are Cohen reals over any $L[z]$ but not comeager many, $PSP_{meager}(\Sigma_2^1)$ is true although Σ_2^1 sets do not necessarily have the Baire property, so one cannot use Mycielski's theorem to prove so.

The last section elaborates on ideals generated by classes of a given equivalence relation E – which we denote by I_E :

Theorem 1.12. *Let E be a Π_1^1 equivalence relation. Then $I = I_E$ is proper.*

Theorem 1.13. *If for every orbit equivalence relation E , \mathbb{P}_{I_E} is proper, then the Vaught conjecture is true.*

1.2. Borel Canonization of Analytic Equivalence Relations. The following problem was raised by Kanovei, Sabok and Zapletal in [15]:

Problem 1.14. *Borel canonization of analytic equivalence relations with Borel classes:* Given an analytic equivalence relation E on a Polish space X , all of its classes Borel, and a proper σ -ideal I , does there exist an I -positive Borel set B such that E restricted to B is Borel?

That problem is strongly related to the main result of this paper via the following celebrated theorem due to Silver:

Theorem 1.15. *(Silver) Let E be a coanalytic equivalence relation on a Polish space X . Then either E has countably many classes, or it has perfectly many classes.*

Let I be a proper σ -ideal, and let E be an analytic equivalence relation with Borel I -small classes. Assume a positive answer to problem 1.14, and fix B a Borel I -positive set such that $E \upharpoonright_B$ is Borel. B must intersect uncountably many classes, and Silver's theorem then provides a perfect set of pairwise inequivalent elements. We have thus proved the following:

Proposition 1.16. *A positive answer to problem 1.14 implies that analytic equivalence relations with Borel I -small classes for I proper must have perfectly many classes.*

That was our original motivation to consider the problems discussed in this paper. However, since the consequence of the positive answer to problem 1.14 turned out to be a theorem of ZFC , it hasn't shed new light on the problem of Borel canonization, which is still open.

1.3. Preliminaries. The basics of universally Baire sets can be found in [6] or the relevant chapter in [13]. Forcing with ideals is thoroughly covered in [21]. [12] contains all generic absoluteness results used along the paper.

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2. UNIVERSALLY BAIRE EQUIVALENCE RELATIONS WITH I -SMALL CLASSES

In the following section we prove:

Theorem 2.1. *Let E be a universally Baire equivalence relation, and I a proper σ -ideal. If all E -classes are I -small, then E has perfectly many classes.*

Corollary 2.2. *Let E be an analytic equivalence relation, and I a proper σ -ideal. If all E -classes are I -small, then E has perfectly many classes. In other words, for any proper σ -ideal I , $PSP_I(\Sigma_1^1)$ is true.*

Remark 2.3. The reader interested only in analytic equivalence relations can avoid using the universally Baire definition of E and rely on analytic absoluteness or Shoenfield's absoluteness instead. For example, analytic equivalence relations remain equivalence relations in all generic extensions because of Shoenfield's absoluteness.

We begin by describing an absoluteness property of universally Baire equivalence relations which will play a central role in the proof of theorem 2.1:

Proposition 2.4. *Let E be a universally Baire equivalence relation. Then E remains an equivalence relation in generic extensions of the universe.*

Proof. For a forcing notion \mathbb{P} , fix trees $T, S \subseteq (\omega \times \omega \times \kappa)$ such that $E = p[T]$ and $\sim E = p[S]$ in \mathbb{P} -generic extensions of the universe. For $t \in \kappa^{<\omega}$, $(t)_0$ and $(t)_1$ denote 2 sequences of length $|t|$ given by some bijection of $\kappa^{<\omega}$ and $(\kappa^{<\omega})^2$. Similarly for $(t)_0, (t)_1, (t)_2$.

We define trees T_r, T_s, T_t whose well foundedness is equivalent to reflexivity, symmetry and transitivity of E , respectively:

$$\begin{aligned} (s, t) \in T_r &\Leftrightarrow (s, s, t) \in S. \\ (s_1, s_2, t) \in T_s &\Leftrightarrow ((s_1, s_2, (t)_0) \in T) \wedge ((s_2, s_1, (t)_1) \in S). \\ (s_1, s_2, s_3, t) \in T_t &\Leftrightarrow ((s_1, s_2, (t)_0) \in T) \wedge ((s_2, s_3, (t)_1) \in T) \wedge ((s_1, s_3, (t)_2) \in S). \end{aligned}$$

Absoluteness of well foundedness of trees concludes the proof. \square

The following lemma is based on [7], theorem 3.4. We will say that \mathbb{P} *adds a new class* if \mathbb{P} adds a real not equivalent to any ground model real:

Lemma 2.5. *Let \mathbb{P} be a proper forcing notion, and E a universally Baire equivalence relation. If \mathbb{P} adds a new class, then E has perfectly many classes.*

Proof. Consider the product $\mathbb{P} \times \mathbb{P}$, and let τ be a name for a real that is not equivalent to any ground model real. We denote by τ_l and τ_r the “left” and “right” names of that real, respectively.

Claim 2.6. For every condition p , $(p, p) \Vdash \neg(\tau_l E \tau_r)$.

Given the claim, pick θ large enough and $M \preceq H_\theta$ a countable elementary submodel containing all the necessary information. We construct a perfect tree $\langle p_s : s \in 2^{<\omega} \rangle$ of conditions of \mathbb{P} such that:

- (1) $p_{s \smallfrown i} \leq p_s$.
- (2) p_s determines at least the first $|s|$ elements of τ .
- (3) For $f \in 2^\omega : \langle p_{f \upharpoonright n} : n \in \omega \rangle$ generate a \mathbb{P} -generic filter over M .
- (4) For $f, g \in 2^\omega : \langle (p_{f \upharpoonright n}, p_{g \upharpoonright n}) : n \in \omega \rangle$ generate a $\mathbb{P} \times \mathbb{P}$ -generic filter over M .
- (5) $(p_{s \smallfrown 0}, p_{s \smallfrown 1}) \Vdash \neg(\tau_l E \tau_r)$.

The construction is inductive. Fix $\langle D_n : n \in \omega \rangle$ an enumeration of the dense open subsets of \mathbb{P} that belong to M , and $\langle D_n^* : n \in \omega \rangle$ an enumeration of the dense open subsets of $\mathbb{P} \times \mathbb{P}$ that belong to M . To construct the $(n+1)$ 'th level of the tree, first extend all p_s of level n to

$$(p_{s \smallfrown 0}, p_{s \smallfrown 1}) \Vdash \neg(\tau_l E \tau_r).$$

Then extend all elements of the new level so that they will belong to D_n , and extend all pairs of elements of the new level so that they will belong to D_n^* . A final extension of the new level will guarantee condition (2) as well.

For $f \in 2^\omega$, let τ_f be the realization of τ by the generic filter generated by $\langle p_{f \upharpoonright n} : n \in \omega \rangle$. The function $f \rightarrow \tau_f$ is continuous, by (2). Using (5), if $f \neq g$ and s is such that $f \supseteq s \smallfrown 0$ and $g \supseteq s \smallfrown 1$, then $(p_{s \smallfrown 0}, p_{s \smallfrown 1})$ is in the generic filter adding τ_f and τ_g , and hence

$$M[G_0][G_1] \models \neg(\tau_f E \tau_g).$$

Since E is universally Baire, $\mathbb{V} \models \neg(\tau_f E \tau_g)$, and E has perfectly many classes. \square

Proof. (of the claim) Assume otherwise, and let $p \in \mathbb{P}$ be such that $(p, p) \Vdash \tau_l E \tau_r$. Pick θ large enough and $M \preceq H_\theta$ a countable elementary submodel containing all the necessary information, and in particular

$p \in M$. The idea will be to consider one M -generic filter which is in \mathbb{V} , and another filter which is generic over both \mathbb{V} and M – where we use properness to guarantee its existence.

So first, let $q \leq p$ be (M, p) -generic. Let

$$p \in G_0 \in \mathbb{V}$$

be a generic filter over M , and $q \in G_1$ a generic filter over \mathbb{V} . Then G_1 is M -generic as well (to be precise – its intersection with $\mathbb{P} \cap M$ is M -generic), and we may find

$$G_2 \in \mathbb{V}[G_1]$$

such that $p \in G_2 \subseteq \mathbb{P} \cap M$ and G_2 is generic over both countable models $M[G_0]$ and $M[G_1]$. Then $G_0 \times G_2$ and $G_1 \times G_2$ are both generic over M and contain (p, p) . It follows that

$$M[G_0][G_2] \models \tau_{G_0} E \tau_{G_2}$$

$$M[G_1][G_2] \models \tau_{G_1} E \tau_{G_2},$$

and using the universally Baire definition:

$$\mathbb{V}[G_1] \models (\tau_{G_0} E \tau_{G_2}) \wedge (\tau_{G_1} E \tau_{G_2}).$$

Since by proposition 2.4 E is still an equivalence relation in $\mathbb{V}[G_1]$,

$$\mathbb{V}[G_1] \models \tau_{G_0} E \tau_{G_1}.$$

Since $\tau_{G_0} \in \mathbb{V}$, we conclude that τ_{G_1} does belong to a ground model equivalence class, although G_1 is \mathbb{V} -generic and τ is a name of a new class. That is a contradiction. \square

Corollary 2.7. *Let \mathbb{P} be a proper forcing notion adding a real, and E a universally Baire equivalence relation. Then \mathbb{P} adds a new class if and only if E has perfectly many classes.*

Proof. One direction is the previous lemma. For the other, note that when a new real is added to the universe, a new real is added to every perfect set of the universe. If B is a Borel set disjoint of some class $[z]_E$, it remains so in \mathbb{P} -generic extensions – since for a universally Baire set, being empty is absolute between generic extensions. We will show that a perfect set of pairwise E inequivalent elements remains such in a \mathbb{P} -generic extension. Hence the new real in the perfect set P has no choice but to belong to a new E -class.

Indeed, given P a perfect tree of pairwise E inequivalent elements, there exists a tree T_P whose well foundedness is equivalent to the pairwise inequivalence of the branches of P :

$$(s_1, s_2, t) \in T_P \Leftrightarrow ((s_1, s_2) \in P) \wedge ((s_1, s_2, (t)_0) \in T) \wedge ((s_1, s_2, (t)_1) \in I)$$

where I is a tree such that I_{xy} is well founded if and only if $x = y$. \square

Proof. (of theorem 2.1) Assume otherwise – E does not have perfectly many classes. Hence by lemma 2.5, forcing with \mathbb{P}_I does not add a new class. Fix $z \in \mathbb{V}$ and $B \in \mathbb{P}_I$ such that

$$B \Vdash x_{gen} \in [z],$$

where x_{gen} stands for the generic real added by \mathbb{P}_I . Let M be an elementary submodel of the universe containing z and all the relevant information. Let $x \in B$ be M -generic. Then $M[x] \models x E z$, and using the universally Baire definition of $[z]$ we know that $\mathbb{V} \models x E z$. We have thus shown that the M -generics in B are all equivalent to z – and in particular $[z]$ is I -positive, contradicting our assumption. \square

Remark 2.8. Corollary 2.7 is interesting in its own but not needed for the proof of theorem 2.1.

3. Δ_2^1 EQUIVALENCE RELATIONS WITH I -SMALL CLASSES

In general, Δ_2^1 equivalence relations can have I -small classes without having perfectly many classes:

Theorem 3.1. [4] *In L , there is a countable Δ_2^1 equivalence relation that does not have perfectly many classes.*

Proof. In L , consider the following equivalence relation:

$$xEy \Leftrightarrow (\forall \alpha \text{ admissible } x \in L_\alpha \Leftrightarrow y \in L_\alpha).$$

Since the constructibility rank of x and the admissibility of ordinals are decided by a countable model and by all countable models, E is a Δ_2^1 equivalence relation. All E classes are countable, since all L'_α s are. We will show that any perfect tree T must have two equivalent elements.

Let $T \in L$ be perfect, and let α be such that $T \in L_\alpha$. Let β be the first admissible ordinal greater than α such that L_β has a real not in L_α . Using [4] fact 9.5, L_α is countable in L_β . Since T has uncountably many branches in L_β , there must be

$$x \neq y \in [T] \cap L_\beta$$

that are not in L_α . It follows that x and y are equivalent. \square

Corollary 3.2. *If $\mathbb{R} = \mathbb{R}^{L[z]}$ for some $z \in \mathbb{R}$, then for any σ -ideal I , $\neg PSP_I(\Delta_2^1)$.*

Proof. A relativization of the above argument. \square

We turn now to the positive results involving Δ_2^1 equivalence relations.

A set A is *provably* Δ_2^1 if the equivalence of the Σ_2^1 and the Π_2^1 definitions is a theorem of ZFC , which is: there are a Σ_2^1 formula $\Phi(x)$ and a Π_2^1 formula $\Psi(x)$ such that

$$ZFC \vdash \forall x : \Phi(x) \leftrightarrow \Psi(x)$$

and Φ is a definition of A . A set A is *provably* Δ_2^1 (boldface) if there is a parameter z and formulas $\Phi(x, z)$, $\Psi(x, z)$ which are Σ_2^1 and Π_2^1 , respectively, such that all ZFC models with the parameter z satisfy

$$\forall x : \Phi(x, z) \leftrightarrow \Psi(x, z).$$

Note that the above formula is $\Pi_3^1(z)$.

Corollary 3.3. (of theorem 2.1) *Let E be a provably Δ_2^1 equivalence relation, and I a proper ideal. If all E -classes are I -small, then E has perfectly many classes. In other words, $PSP_I(\text{provably } \Delta_2^1)$ for any proper σ -ideal I .*

Proof. It is easy to see that provably Δ_2^1 sets are universally Baire. In fact, any set with a Δ_2^1 definition preserved in generic extensions is a universally Baire set. \square

Hence provably Δ_2^1 equivalence relations do not present a new challenge. The rest of the section is devoted to the case of a general Δ_2^1 equivalence relation.

We say that a forcing \mathbb{P} has Π_3^1 - \mathbb{P} -absoluteness if \mathbb{V} and $\mathbb{V}^\mathbb{P}$ agree on Π_3^1 statements with parameters in \mathbb{V} . For most forcing notions \mathbb{P} , Π_3^1 - \mathbb{P} -absoluteness is independent of ZFC .

Theorem 3.4. *Let I be a proper σ -ideal, and assume Π_3^1 - \mathbb{P}_I -absoluteness. Then $PSP_I(\Delta_2^1)$.*

The proof is a variant of the proof of theorem 2.1. We restate the lemmas and corollary in the new context and indicate the main differences in the proofs.

Proof. Let E be a Δ_2^1 equivalence relation with I -small classes. We may assume E is lightface Δ_2^1 . Fix $\Phi(x, y)$ a Σ_2^1 formula and $\Psi(x, y)$ a Π_2^1 formula, both defining E , so that

$$\mathbb{V} \models \forall x, y : \Phi(x, y) \Leftrightarrow \Psi(x, y).$$

Because of Π_3^1 - \mathbb{P} -absoluteness, the Σ_2^1 and Π_2^1 definitions will coincide in all generic extensions of \mathbb{V} . In particular, E defined by Φ and Ψ will continue being an equivalence relation in generic extensions – using the above observations and Shoenfield’s absoluteness.

Lemma 3.5. *Let \mathbb{P} be a proper forcing notion, and E a Δ_2^1 equivalence relation. Assume Π_3^1 - \mathbb{P} -absoluteness. Then if \mathbb{P} adds a new E class, then E has perfectly many classes.*

Proof. Consider the product $\mathbb{P} \times \mathbb{P}$, and let τ , τ_l and τ_r be as in lemma 2.5. Φ and Ψ are as above.

Claim 3.6. For every condition p , $(p, p) \Vdash \Phi(x, y)$, which in light of the above is the same as $(p, p) \Vdash \Psi(x, y)$.

Given the claim, pick θ large enough and $M \preceq H_\theta$ a countable elementary submodel containing all the necessary information. We construct a perfect tree $\langle p_s : s \in 2^{<\omega} \rangle$ of conditions of \mathbb{P} such that:

- (1) $p_{s \smallfrown i} \leq p_s$.
- (2) p_s determines at least the first $|s|$ elements of τ .
- (3) For $f \in 2^\omega : \langle p_{f \upharpoonright n} : n \in \omega \rangle$ generate a \mathbb{P} -generic filter over M .
- (4) For $f, g \in 2^\omega : \langle (p_{f \upharpoonright n}, p_{g \upharpoonright n}) : n \in \omega \rangle$ generate a $\mathbb{P} \times \mathbb{P}$ -generic filter over M .
- (5) $(p_{s \smallfrown 0}, p_{s \smallfrown 1}) \Vdash \neg \Psi(\tau_l, \tau_r)$.

From here we continue just as in the proof of lemma 2.5, with analytic absoluteness enough to complete the proof. \square

Proof. (of the claim) Exactly as in lemma 2.5, with xEy replaced by $\Phi(x, y)$, till the point we have

$$M[G_0][G_2] \models \Phi(\tau_{G_0}, \tau_{G_2})$$

$$M[G_1][G_2] \models \Phi(\tau_{G_1}, \tau_{G_2}).$$

By analytic absoluteness:

$$\mathbb{V}[G_1] \models \Phi(\tau_{G_0}, \tau_{G_2}) \wedge \Phi(\tau_{G_1}, \tau_{G_2}).$$

As previously mentioned, Φ remains an equivalence relation in $\mathbb{V}[G_1]$, and so

$$\mathbb{V}[G_1] \models \Phi(\tau_{G_0}, \tau_{G_1}).$$

Since $\tau_{G_0} \in \mathbb{V}$, we conclude that τ_{G_1} does belong to a ground model equivalence class, although G_1 is \mathbb{V} -generic and τ is a name of a new class. That is a contradiction. \square

Note that in the proof we have used both the Σ_2^1 and the Π_2^1 definitions.

Corollary 3.7. *Let \mathbb{P} be a proper forcing notion adding a real, and E a Δ_2^1 equivalence relation. Assume Π_3^1 - \mathbb{P} -absoluteness. Then \mathbb{P} adds a new class if and only if E has perfectly many classes.*

Proof. One direction is the above proof. The other is similar to the proof of corollary 2.7, where absoluteness arguments now follow of Shoenfield’s theorem. \square

We can now complete the proof of theorem 3.4 in the same way we have proved theorem 2.1 – here $M[x] \models xEz$ implies $\mathbb{V} \models xEz$ follows of analytic absoluteness. \square

Together with [12], we have shown:

Theorem 3.8. *The following are equivalent:*

- (1) $PSP_{countable}(\Delta_2^1)$.
- (2) For z real, $\mathbb{R}^{L[z]} \neq \mathbb{R}$.

Proof. (1) \Rightarrow (2) is theorem 3.1. (2) \Rightarrow (1) follows from theorem 3.4 since Ikegami has shown in [12] that (2) is equivalent to Π_3^1 -Sacks-absoluteness. \square

Remark 3.9. For the case of the meager and null ideal, we have:

If for any z real, there is a Cohen (random) real over $L[z]$, then $PSP_{meager(null)}(\Delta_2^1)$.

To see why, use theorem 3.4 together with the fact that existence of Cohen (random) reals over any $L[z]$ is equivalent to Π_3^1 -Cohen (random) absoluteness.

However, this is not a new result – it follows from Mycielski’s theorems 1.1 and 1.2 together with Ihoda-Shelah theorem on the Baire property (and Lebesgue measurability) of Δ_2^1 sets.

Remark 3.10. In [12] theorem 4.3 it is proved that for a wide class of σ -ideals, “ Π_3^1 - \mathbb{P}_I -absoluteness” is equivalent to “all Δ_2^1 sets are \mathbb{P}_I -Baire”. A set is universally Baire if and only if it is \mathbb{P} -Baire for every forcing notion \mathbb{P} .

Using the above terminology and referring to ideals to which [12] theorem 4.3 applies, a result of section 2 is that if every Δ_2^1 set is \mathbb{P} -Baire for any \mathbb{P} , and I is any proper ideal, then $PSP_I(\Delta_2^1)$. Section 3 shows that if for a given proper ideal I , every Δ_2^1 set is \mathbb{P}_I -Baire, then $PSP_I(\Delta_2^1)$. In that sense, section 3 provides a “local” version of the result of section 2.

4. Σ_2^1 AND Π_2^1 EQUIVALENCE RELATIONS WITH MEAGER CLASSES

In this section we focus our attention on the meager ideal.

Note that until now, we have not given any new result on equivalence relations with meager classes. Considering section 2, for example, if E is universally Baire with meager classes, then it has the Baire property, and then Mycielski’s theorem 1.1 is valid. Regarding section 3, whenever forcing with non-meager Borel sets has Π_3^1 generic absoluteness, then Δ_2^1 sets have the Baire property – and yet again theorem 1.1 applies. The following section introduces a case in which Mycielski’s theorem does not apply and we can still obtain the desired perfect set property for equivalence relations with meager classes.

For the following recall that the existence of Cohen reals over $L[z]$ for any real z is equivalent to Π_3^1 -Cohen absoluteness. Unless otherwise noted, $I = meager$.

Theorem 4.1. *If for any real z there is a Cohen real over $L[z]$ then*

$$PSP_{meager}(\Sigma_2^1)$$

and

$$PSP_{meager}(\Pi_2^1 \text{ with Borel classes}).$$

Lemma 4.2. *Assume that for any real z there is a Cohen real over $L[z]$. Let E be a Σ_2^1 equivalence relation or a Π_2^1 equivalence relation. If the \mathbb{P}_I -generic belongs to a new E -class then E has perfectly many classes.*

Proof. The Π_3^1 - \mathbb{P}_I -absoluteness guarantees that E will remain an equivalence relation in \mathbb{P}_I -generic extensions.

For ease of notation, we assume E is lightface Σ_2^1 or Π_2^1 . Consider the product $\mathbb{P}_I \times \mathbb{P}_I$, and let τ be a name for the \mathbb{P}_I -generic, τ_l a name for the \mathbb{P}_I -generic added by the left \mathbb{P}_I and τ_r a name for the generic added by the right one.

Claim 4.3. For every condition p , $(p, p) \not\Vdash (\tau_l E \tau_r)$.

Assume the claim. When E is Π_2^1 , the proof continues in exactly the same way it did in the previous section. For $E \Sigma_2^1$, we will construct a perfect tree $\langle p_s : s \in 2^{<\omega} \rangle$ of elements of \mathbb{P}_I such that:

- (1) $p_{s \smallfrown i} \leq p_s$.
- (2) p_s determines at least the first $|s|$ elements of τ .
- (3) For $f \in 2^\omega : \langle p_{f \upharpoonright n} : n \in \omega \rangle$ generate a \mathbb{P}_I -generic filter **over** L .
- (4) For $f, g \in 2^\omega : \langle p_{f \upharpoonright n}, p_{g \upharpoonright n} \rangle : n \in \omega \rangle$ generate a $\mathbb{P}_I \times \mathbb{P}_I$ -generic filter **over** L .
- (5) $L \models (p_{s \smallfrown 0}, p_{s \smallfrown 1}) \Vdash \neg(\tau_l E \tau_r)$, which in our case of Cohen forcing is just the same as

$$(p_{s \smallfrown 0}, p_{s \smallfrown 1}) \Vdash \neg(\tau_l E \tau_r).$$

For the construction we rely on the following fact:

Fact 4.4. ([3] 1.1) *If there is a Cohen real over $L[z]$ then there is a perfect set of $\mathbb{P}_I \times \mathbb{P}_I$ generics over $L[z]$.*

All we need to do now is to refine the perfect tree of the $\mathbb{P}_I \times \mathbb{P}_I$ generics. Shoenfield's absoluteness completes the proof: if $L[x][y] \models \neg(\tau_l E \tau_r)$ then $\mathbb{V} \models \neg(\tau_l E \tau_r)$. \square

Proof. (of the claim) If E is Σ_2^1 , the proof of the previous section works. We give the proof for $E \Pi_2^1$. The fact that a Cohen generic over \mathbb{V} is generic over all inner models of \mathbb{V} is used over and over again.

Assume the claim fails, and let $p \in \mathbb{P}_I$ be such that $(p, p) \Vdash \tau_l E \tau_r$. Let

$$p \in G_0 \in \mathbb{V}$$

be a generic filter over L – there is one, since when a Cohen real over L exists, every non meager set has one. Let $p \in G_1$ be a generic filter over \mathbb{V} , and let G_2 be generic over $\mathbb{V}[G_1]$ such that $p \in G_2$. Then $G_0 \times G_2$ and $G_1 \times G_2$ are both generic over L and contain (p, p) . It follows that

$$L[G_0][G_2] \models \tau_{G_0} E \tau_{G_2}$$

$$L[G_1][G_2] \models \tau_{G_1} E \tau_{G_2}.$$

By Shoenfield's absoluteness, these statements are still true in $\mathbb{V}[G_1][G_2]$. Recall that \mathbb{P}_I and $\mathbb{P}_I \times \mathbb{P}_I$ are equivalent, therefore Π_3^1 absoluteness still applies for $\mathbb{P}_I \times \mathbb{P}_I$ and E is transitive in $\mathbb{V}[G_1][G_2]$. Using absoluteness again we see that

$$\mathbb{V}[G_1] \models \tau_{G_0} E \tau_{G_1}.$$

But $\tau_{G_0} \in \mathbb{V}$, whereas τ_{G_1} is generic over \mathbb{V} , so τ_{G_1} belongs to a ground model equivalence class – which is a contradiction. \square

Proof. (of theorem 4.1) For $E \Sigma_2^1$, exactly as in the previous section. For $E \Pi_2^1$, one uses the additional assumption that the classes are Borel, in which case the \mathbb{P}_I -generic must belong to a new E -class. \square

Theorem 4.1 is indeed stronger than Mycielski's theorem 1.1 in the following sense – in a universe in which there are Cohen reals over any $L[z]$ but not comeager many, $PSP_{meager}(\Sigma_2^1)$ is true but Σ_2^1 sets do not necessarily have the Baire property.

Remark 4.5. We conjecture that $PSP_{meager}(\Sigma_2^1)$ is equivalent to the existence of Cohen generics over $L[z]$ for any real z .

5. σ -IDEALS GENERATED BY EQUIVALENCE RELATIONS

Given an equivalence relation E , let I_E be the σ -ideal generated by the E -equivalence classes.

Example 5.1. For $x, y \in \omega^\omega$, let

$$xE_{ck}y \Leftrightarrow \omega_1^{ck(x)} = \omega_1^{ck(y)}.$$

Let x_{gen} be the generic real added by forcing with $\mathbb{P}_{I_{E_{ck}}}$. Then $\omega_1^{ck(x_{gen})} \geq \omega_1$, and in particular, $I_{E_{ck}}$ is improper.

Example 5.2. Assume the Vaught conjecture is false, and let (G, X) be a counterexample (G a Polish group and X a Polish space). Let $E = E_G^X$ be the induced equivalence relation, and δ a Hjorth rank associated with the action (see [10, 5]). Recall that for a countable ordinal α ,

$$\mathcal{A}_\alpha = \{x : \delta(x) \leq \alpha\}$$

is Borel and the orbit equivalence relation restricted to \mathcal{A}_α is Borel as well. Silver's theorem now guarantees that \mathcal{A}_α is a **countable** union of equivalence classes – therefore $\mathcal{A}_\alpha \in I_E$. The generic real x_{gen} added by \mathbb{P}_{I_E} must then have rank at least ω_1 , proving the impropriety of I_E .

Theorem 5.3. *Let E be an analytic or coanalytic equivalence relation such that every Borel set intersecting uncountably many classes, has perfectly many classes. Then $I = I_E$ is proper.*

Proof. Pick θ large enough and $M \preceq H_\theta$ a countable elementary submodel, and let $B \in M$ be a Borel I -positive set. We will find a perfect set of pairwise inequivalent elements, all in B and generic over M – therefore proving the properness of I .

Consider the product $\mathbb{P}_I \times \mathbb{P}_I$, and let τ be a name for the generic real. We denote by τ_l and τ_r the “left” and “right” names of the new real, respectively.

Claim 5.4. For every condition B , $(B, B) \not\Vdash \tau_l E \tau_r$.

Proof. Let $B \in \mathbb{P}_I$. Then B intersects uncountably many classes, hence by the assumption it contains a perfect set of pairwise inequivalent elements. It is easy to see that B contains two disjoint perfect sets B_0 and B_1 , both of which of pairwise inequivalent elements, such that their saturations are disjoint. If

$$B_0 \times B_1 \Vdash \neg(\tau_l E \tau_r).$$

the proof of the claim will be completed. Indeed,

$$\mathbb{V} \models \forall x \in B_0 \forall y \in B_1 \neg(xEy),$$

which is a Π_2^1 statement, therefore $\mathbb{V}[G_0][G_1] \models \neg(\tau_l E \tau_r)$. □

We can now fix $M \preceq H_\theta$ a countable elementary submodel and repeat the same construction carried out in the proof of lemma 2.5, resulting in a perfect tree of conditions. The different branches through the tree induce a perfect set P of mutually M -generic elements. For $x \neq y$ in P ,

$$M[x][y] \models \neg(xEy)$$

and absoluteness completes the proof. □

Corollary 5.5. *Let E be a Π_1^1 equivalence relation. Then $I = I_E$ is proper.*

Proof. By Silver's theorem, every coanalytic equivalence relation satisfies the condition of theorem 5.3. \square

Corollary 5.6. *Let E be an analytic equivalence relation, and $I = I_E$. Then \mathbb{P}_I is proper if and only if every Borel set intersecting uncountably many classes, has perfectly many classes. In particular, if for every orbit equivalence relation E , \mathbb{P}_{I_E} is proper, then the Vaught conjecture is true.*

Proof. One direction is corollary 2.2 restricted to a Borel I -positive set. The other is theorem 5.3. \square

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